HOSSAM GHANEM

(12) 7.6 Hyperbolic functions (B)

$\lvert\tanh x\rvert \leq 1$
$\lvert\coth x\rvert \leq 1$

Differentiation
$\frac{d}{dx}\sinh x = \cosh x$
$\frac{d}{dx}\cosh x = \sinh x$
$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$
$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$
$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$

Integration
$\int \sinh x \ dx = \cosh x + C$
$\int \cosh x \ dx = \sinh x + C$
$\int \operatorname{sech}^2 x \, dx = \tanh x + C$
$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$
$\int \operatorname{csch} x \tanh x \, dx = -\operatorname{csch} x + C$
$\int \operatorname{csch}^2 x \, dx = -\coth x + C$

$\cosh^2 x - \sinh^2 x = 1$	$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$	$\sinh 2x = 2 \sinh x \cosh x$
$1 - \operatorname{sech}^2 x = \tanh^2 x$		$\cosh 2x = \cosh^2 x + \sinh^2 x$
$1 + \operatorname{csch}^2 x = \coth^2 x$	$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$	$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$

Example 1 Find y' if $x^3 \sin^{-1}(e^{3x}) + \ln(\cosh y) = xy$ 20 Nov. 2006 A

Solution

$$3x^2 \sin^{-1}(e^{3x}) + x^3 \frac{3e^{3x}}{\sqrt{1-e^{6x}}} + \frac{\sinh y}{\cosh y} y' = y + xy'$$

$$y'[\tanh y - x] = y - 3x^2 \sin^{-1}(e^{3x}) - \frac{3e^{3x}}{\sqrt{1-e^{6x}}}$$

$$y' = \left[y - 3x^2 \sin^{-1}(e^{3x}) - \frac{3e^{3x}}{\sqrt{1-e^{6x}}}\right][\tanh y - x]^{-1}$$

Example 2 Find $\frac{dy}{dx}$ if 8 October 1997

$$\sinh(\sec y) + \ln(e^x + 1) = \cosh y + \sin^{-1} x^2$$

Solution

$$\cosh(\sec y) \cdot \sec y \tan y \, y' + \frac{e^x}{e^x + 1} = \sinh y \; y' + \frac{2x}{\sqrt{1-x^4}}$$

$$y' = [\cosh(\sec y) \cdot \sec y \tan y - \sinh y] = \frac{2x}{\sqrt{1-x^4}} - \frac{e^x}{e^x + 1}$$

$$\frac{dy}{dx} = \left[\frac{2x}{\sqrt{1-x^4}} - \frac{e^x}{e^x + 1}\right][\cosh(\sec y) \cdot \sec y \tan y - \sinh y]^{-1}$$

Example 3 Find $\frac{dy}{dx}$ if 21 March 2007 A

$$y = (\sin x)^{\cosh x} + \pi^{\sec^{-1} x} + e^{\pi}$$

Solution

$$y_1 = (\sin x)^{\cosh x}$$

$$\ln y_1 = \cosh x \ln(\sin x)$$

$$\frac{y'_1}{y_1} = \sinh x \ln(\sin x) + \cosh x \cdot \frac{\cos x}{\sin x}$$

$$y'_1 = (\sin x)^{\cosh x}[\sinh x \ln(\sin x) + \cosh x \cdot \cot x]$$

$$y_2 = \pi^{\sec^{-1} x} + e^{\pi}$$

$$y'_2 = \pi^{\sec^{-1} x} \cdot \frac{1}{x\sqrt{x^2-1}} \ln \pi + 0$$

$$\frac{dy}{dx} = y'_1 + y'_2 = (\sin x)^{\cosh x}[\sinh x \ln(\sin x) + \cosh x \cdot \cot x] + \pi^{\sec^{-1} x} \cdot \frac{1}{x\sqrt{x^2-1}} \ln \pi$$

Example 4 Find $\frac{dy}{dx}$ if $y = \frac{(\tanh x)^{\tan x} \; 3^{\ln(\sin^{-1} x)}}{e^{\sec(x^3)} \sqrt{2-x^3}}$ 22 July 2007

Solution

$$\ln y = \tan x \ln(\tanh x) + \ln(\sin^{-1} x) \ln 3 - \sec(x^3) - \frac{1}{2}\ln(2-x^3)$$

$$\frac{y'}{y} = \sec^2 x \ln(\tanh x) + \tan x \frac{\operatorname{sech}^2 x}{\tanh x} + \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} \ln 3 - \frac{-3x^2}{2(2-x^3)}$$

Example 5 Use logarithmic differentiation to find $\frac{dy}{dx}$ if 24 March 2008 A

$$y = \frac{(\cosh x)^{\sec^{-1} x}}{\sqrt{e^x \ln|x|}}$$

Solution

$$\ln y = \sec^{-1} x \ln(\cosh x) - \frac{1}{2}x - \frac{1}{2}\ln(\ln|x|)$$

$$\frac{y'}{y} = \frac{1}{x\sqrt{x^2-1}} \ln(\cosh x) + \sec^{-1} x \cdot \frac{\sinh x}{\cosh x} - \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{\ln|x|} \cdot \frac{1}{x}$$

$$y' = \left[\frac{(\cosh x)^{\sec^{-1} x}}{\sqrt{e^x \ln|x|}}\right]\left[\frac{1}{x\sqrt{x^2-1}} \ln(\cosh x) + \sec^{-1} x \tanh x - \frac{1}{2} - \frac{1}{2x \ln|x|}\right]$$

Example 6 Find $\frac{dy}{dx}$ if $y = \frac{\sqrt[3]{\cosh x + \tanh x}\,(1 + \sin(e^{-2x}))}{|x - \cosh x|^{x^3}}$ 25 April 2008

Solution

$$\ln y = \frac{1}{3}\ln(\cosh x + \tanh x) + \ln(1 + \sin(e^{-2x})) - x^3 \ln|x - \cosh x|$$

$$\frac{y'}{y} = \frac{\sinh x + \operatorname{sech}^2 x}{3(\cosh x + \tanh x)} + \frac{-2e^{-2x} \cos e^{-2x}}{1 + \sin(e^{-2x})} - 3x^2 \ln|x - \cosh x| - x^3 \cdot \frac{1 - \sinh x}{x - \cosh x}$$

$$y' = y\left(\frac{\sinh x + \operatorname{sech}^2 x}{3(\cosh x + \tanh x)} + \frac{-2e^{-2x} \cos e^{-2x}}{1 + \sin(e^{-2x})} - 3x^2 \ln|x - \cosh x| - x^3 \cdot \frac{1 - \sinh x}{x - \cosh x}\right)$$

Example 7 Evaluate the integral $\int (\coth x)\ \ln(\sinh x)\ dx$ 28 April 2009 A

Solution

$$t = \ln(\sinh x) \qquad dt = \frac{\cosh x}{\sinh x}\, dx \qquad dt = \coth x\ dx$$

$$I = \int (\coth x)\ \ln(\sinh x)\ dx = \int \ln(\sinh x) \cdot (\coth x) dx = \int t\ dt = \frac{1}{2}\, t^2 + c = \frac{1}{2}\, (\ln(\sinh x))^2 + c$$

Example 8 Evaluate the following integral $\int \frac{1 - e^{-2x}}{2e^{-x}} \cosh x\ dx$ 27 Nov. 2008 A

Solution

$$I = \int \frac{1 - e^{-2x}}{2e^{-x}} \cosh x\ dx = \int \frac{e^x(1 - e^{-2x})}{e^x \cdot 2e^{-x}} \cosh x\ dx = \int \frac{e^x - e^{-x}}{2} \cosh x\ dx = \int \sinh x \cosh x\ dx$$

Let $t = \sinh x$, $dt = \cosh x\ dx$

$$I = \int t\ dt = \frac{1}{2}\, t^2 + c = \frac{1}{2}\sinh^2 x + c$$

Example 9 Evaluate the following integral $\int \sqrt{\cosh x - 1}\, dx$ 15 July 2003 A

Solution

$$I = \int \sqrt{\cosh x - 1}\, dx = \int \frac{\sqrt{\cosh x - 1} \cdot \sqrt{\cosh x + 1}}{\sqrt{\cosh x + 1}}\, dx = \int \frac{\sqrt{\cosh^2 x - 1}}{\sqrt{\cosh x + 1}}\, dx$$

$$= \int \frac{\sqrt{\sinh^2 x}}{\sqrt{\cosh x + 1}}\, dx = \int \frac{\sinh x}{\sqrt{\cosh x + 1}}\, dx$$

Let $t = \cosh x + 1$ $dt = \sinh x\ dx$

$$I = \int \frac{1}{\sqrt{t}}\, dt = 2\sqrt{t} + c = 2\sqrt{\cosh x + 1} + c$$

Example 10 Evaluate the integral $\int \frac{\tanh x}{\sqrt{4\cosh^2 x - 1}} dx.$ 24 March 2008 A

Solution

$$I = \int \frac{\tanh x}{\sqrt{4\cosh^2 x - 1}} dx. = \int \frac{\sinh x}{\cosh x \sqrt{4\cosh^2 x - 1}} dx.$$

Let $t = \cosh x$ $dt = \sinh x\ dx$

$$I = \int \frac{1}{t\sqrt{4t^2 - 1}}\, dt = \frac{1}{2}\int \frac{1}{t\sqrt{t^2 - \frac{1}{4}}}\, dt = \frac{1}{2} \cdot 2\sec^{-1} 2t + c = \sec^{-1}(2\cosh x) + c$$

Example 12 Evaluate the integral $\int \frac{5}{\tanh x \sqrt{\cosh^2 x - 5}}\, dx$ 22 July 2007

Solution

$$I = \int \frac{5}{\tanh x \sqrt{\cosh^2 x - 5}}\, dx = \int \frac{5\cosh x}{\sinh x \sqrt{\cosh^2 x - 5}}\, dx = \int \frac{5\cosh x}{\sinh x \sqrt{\sinh^2 x - 4}}\, dx$$

Let $t = \sinh x$ $dt = \cosh x\ dx$

$$I = 5\int \frac{1}{t\sqrt{t^2 - 4}}\, dt = 5 \cdot \frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) + c = \frac{5}{2}\sec^{-1}\left(\frac{\sinh x}{2}\right) + c$$

Example 13 Evaluate the following integral $\int x \tanh(\ln x)\ dx$ 25 April 2008

Solution

$$I = \int x \tanh(\ln x)\ dx = \int x \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}\, dx = \int x\left(\frac{x - \frac{1}{x}}{x - \frac{1}{x}}\right) dx = \int x\left(\frac{x^2 - 1}{x^2 + 1}\right) dx$$

$$= \int x(\ \ )\, dx = \int x\left(1 - \frac{2}{x^2 + 1}\right) dx = \int x - \frac{2x}{x^2 + 1}\, dx = \frac{1}{2}x^2 - \ln(x^2 + 1) + c$$

Example 14 Let $y = A\sinh(mx) + B\cosh(mx)$ 27 November 2008 A

where A , B and m are constants. Show that $y'' = m^2y$

Solution

$y\backslash = Am\ \cosh\ (mx) + Bm\ \sinh\ (mx)$
$y'' = Am^2\ \sinh\ (mx) + Bm^2\ \cosh\ (mx)$
$y'' = m^2[A\ \sinh\ (mx) + B\ \cosh\ (mx)]$
$y'' = m^2y$

Homework

1 Use logarithmic differentiation to find y' given 26 July 2008 A

$$y = \frac{\sinh x \tan^{-1} x}{\sqrt[3]{\cosh x \cos^{-1} x}}$$

2 Find $\frac{dy}{dx}$ if $y = 3^{\sec^{-1} x}(\cosh x)^x$ 8 October 1997

3 Use logarithmic differentiation to find $\frac{dy}{dx}$ if 21 January 2004 A

$$y = \frac{|x+1|^{\sin^{-1} x}}{\left(e^{-x^2} + \operatorname{sech} x\right)\sqrt[3]{2^{-x} + x^2}}$$

4 Use logarithmic differentiation to find $\frac{dy}{dx}$ if 22 June 2004 A

$$y = \frac{\sqrt[5]{x+1}\,(1+x^2)^x e^{\left(x^2 + \tan^{-1} x\right)}}{\cosh x + \operatorname{sech} x}$$

5 Find $\frac{dy}{dx}$ if $y = \frac{e^{\operatorname{sech} x}\sqrt{x^2-2}}{\ln x\,(\sin^{-1} x)^x}$ 7 July 1997

6 Use logarithmic differentiation to find y' if 29 January 2007 A

$$y = \frac{(\tanh x)^{\sin x}}{\sqrt{3^x - x}}, \quad x > 0$$

7 Use logarithmic differentiation to find $\frac{dy}{dx}$ where 25 August 2005 A

$$y = \frac{x^{\ln\cosh x}\ 3^{\sin^{-1}\sqrt{x}}}{\sqrt[3]{1-2x}}$$

Homework

8	Find y' if y is defined by the formula: $2^y = 8^{\sin^{-1} x} \sinh 2^x$	14 March 2002
9	Find $\frac{dy}{dx}$ if $y = \left(\log_5(\sqrt{x}+1)\right)^{\tanh x}$	13 March 2001 A
10	Find $\frac{dy}{dx}$ if $x \sin^{-1} y + x\, e^{\cosh y} - xy = x + y$	2 March 1993
11	Find $\frac{dy}{dx}$ if $y^2 + \ln\lvert y.4\rvert + xy + \tanh x = 0$	5 October 1996
12	Find $\frac{dy}{dx}$ if $x \sin^{-1} y + x\, e^{\cosh y} - xy = x + y$	2 March 1993
13	Find $\frac{dy}{dx}$ if $y^2 + \ln\lvert y.4\rvert + xy + \tanh x = 0$	5 October 1996
14	Find $\frac{dy}{dx}$ if $\sinh(e^{xy}) + \ln(\cosh y) = x$	10 March 1999
15	Find $\frac{dy}{dx}$ if $\sin^{-1}(x+y) + y \tan^{-1} x + \sinh(xy) = 0$.	13 March 2001 A
16	Find $\frac{dy}{dx}$ if $e^{xy} = \tan^{-1}(\sinh y) + (1+x)^x$	18 July 2005 A
17	Find y' if $x^3 \sin^{-1}(e^{3x}) + \ln(\cosh y) = xy$	20 Nov. 2006 A
18	Evaluate the following integrals $\int \frac{\operatorname{sech}^2 x\ dx}{1 + 3\ \tanh x}$	11 October 1999
19	Evaluate the following integrals $\int \frac{\cosh x\, dx}{\sqrt{1 - \sinh^2 x}}$	2 March 1993
20	Evaluate the following integrals $\int \frac{\sinh x}{5 + \sinh^2 x} dx.$	9 October 1998 19 March 2006 A
21	Evaluate the following integrals $\int \frac{\sinh x \cosh x}{2 + \sinh^2 x}\ dx$	20 Nov. 2006 A
22	Evaluate the following integrals $\int \frac{\cosh x}{4\cosh^2 x - 3\sinh^2 x}\ dx$	14 March 2002
23	Evaluate the following integrals $\int \frac{\tanh x}{\sqrt{3\sinh^2 x + \cosh^2 x - 1}} dx$	31 10 July 2010
24	Evaluate the following integrals $\int \frac{\sinh x}{e^{\cosh x} + e^{-\cosh x}}\ dx$	32 Oct. 31st , 2010 A

Homework

25	(3 pts) Evaluate the following integrals.	$\int \frac{\tanh x}{\sqrt{\cosh^2 x - 16}} dx$	34 July 9, 2011
26	(3 pts) Evaluate the following integrals.	$\int \tanh(\ln(x))\, dx$	34 July 9, 2011
27	Evaluate the following integral	$\int \frac{\cosh x}{4\cosh^2 x + 5} dx$	30 April 11, 2010

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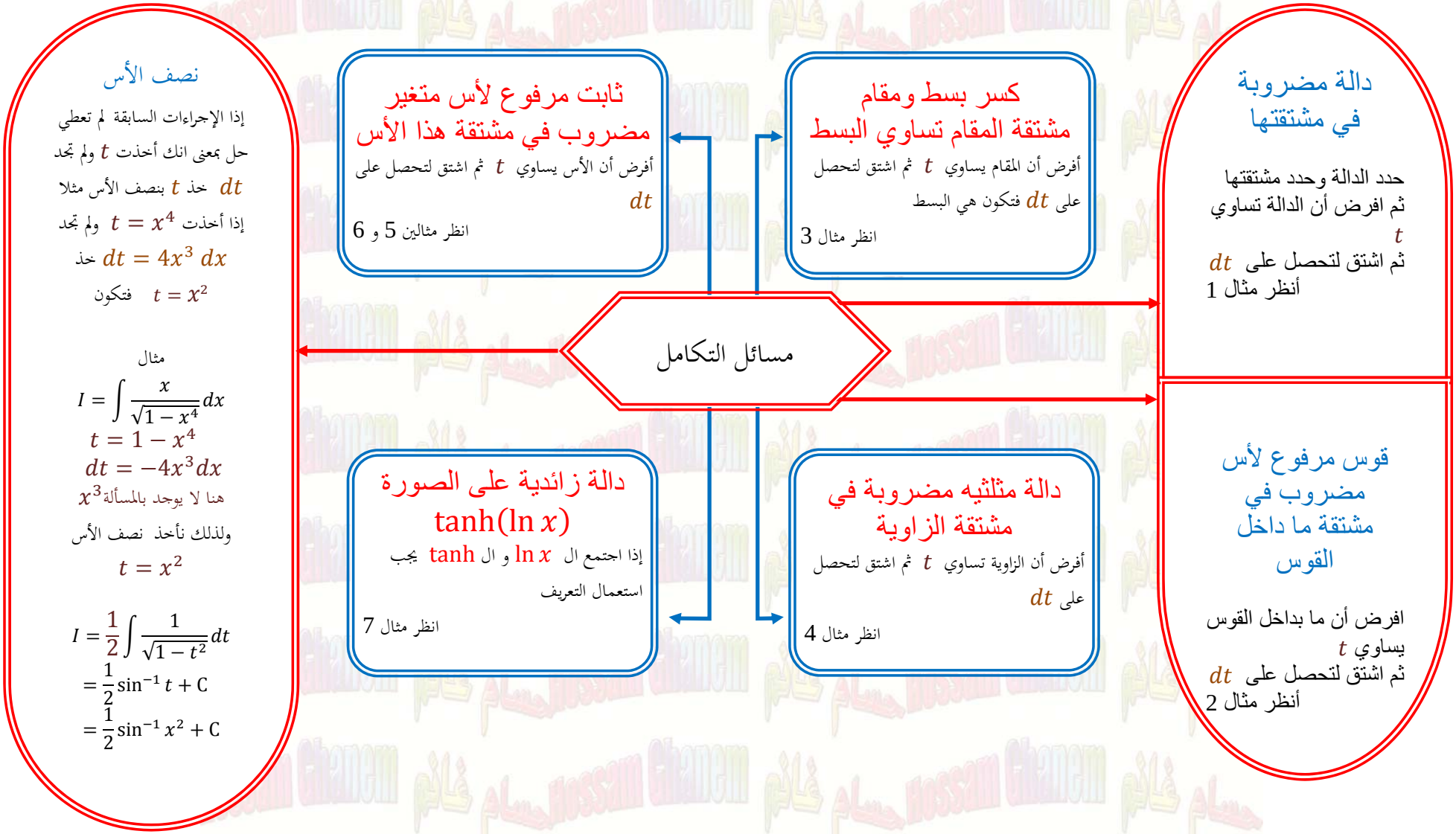

Example 1

$$\int x \sin x^2 \,(1+\cos x^2)dx$$

$$u = 1+\cos x^2$$

$$du = -\sin x^2 \, 2x dx$$

$$\frac{-1}{2}\, du = x \sin x^2 \; dx$$

$$I = \int (1+\cos x^2)\cdot x \sin x^2 \, dx$$

$$= \frac{-1}{2}\int u \; du = \frac{-1}{2}\cdot\frac{1}{2}u^2 + c$$

$$= = \frac{-1}{4}\,(1+\cos x^2)^2 + c$$

Example 2

$$\int \frac{1}{x^3}\left(1+\frac{1}{x^2}\right)^{\frac{5}{3}} dx$$

$$u = 1+\frac{1}{x^2} \qquad u = 1+x^{-2}$$

$$du = -2x^{-3}\, dx$$

$$-\frac{1}{2}\, du = \frac{1}{x^3}\, dx$$

$$I = \int \left(1+\frac{1}{x^2}\right)^{\frac{5}{3}}\cdot\frac{1}{x^3} dx$$

$$= \frac{-1}{2}\int u^{\frac{5}{3}}\, du = \frac{-1}{2}\cdot\frac{3}{8}\, u^{\frac{8}{3}} + c$$

$$= \frac{-3}{16}\left(1+\frac{1}{x^2}\right)^{\frac{8}{3}} + c$$

Example 3

$$\int \frac{e^x+e^{-x}}{e^x-e^{-x}} dx$$

$$I = \int \frac{e^x+e^{-x}}{e^x-e^{-x}} dx$$

$$u = e^x - e^{-x}$$

$$du = (e^x+e^{-x})dx$$

$$I = \int \frac{1}{e^x-e^{-x}}\cdot(e^x+e^{-x})dx$$

$$I = \int \frac{1}{u}\, du = \ln|u| + c$$

$$I = \ln|e^x - e^{-x}| + c$$

Example 4

$$\int \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}} dx$$

$$u = \sqrt[3]{x} \qquad u = x^{\frac{1}{3}}$$

$$du = \frac{1}{3}\, x^{\frac{-2}{3}}\, dx$$

$$3du = \frac{1}{\sqrt[3]{x^2}}\; dx$$

$$I = \int \sin\sqrt[3]{x}\cdot\frac{1}{\sqrt[3]{x^2}} dx$$

$$= 3\int \sin u \, du$$

$$= -3\cos u + c$$

$$= -3\cos\sqrt[3]{x} + c$$

Example 5

$$\int \frac{5^{\cot x}}{\sin^2 x} dx$$

$$u = \cot x \qquad du = -\csc^2 x \, dx$$

$$-du = \frac{1}{\sin^2 x}\, dx$$

$$I = \int 5^{\cot x}\cdot\frac{1}{\sin^2 x} dx$$

$$= -\int 5^u \, du = -5^u\cdot\frac{1}{\ln 5} + c$$

$$= -5^{\cot x}\cdot\frac{1}{\ln 5} + c$$

Example 6

$$\int 2^{-x}6^x dx$$

$$2^{-x}\,6^x = \left(\frac{1}{2}\right)^x 6^x = \left(\frac{6}{2}\right)^x = 3^x$$

$$I = \int 2^{-x}\,6^x\, dx$$

$$= \int 3^x\, dx = \frac{1}{\ln 3}\cdot 3^x + c$$

Example 7

$$I = \int x\tanh(\ln x)\; dx = \int x\frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}}\, dx = \int x\left(\frac{x-\frac{1}{x}}{x-\frac{1}{x}}\right) dx$$

$$= \int x\left(\frac{x^2-1}{x^2+1}\right) dx = \int x\left(\frac{x^2+1-2}{x^2+1}\right) dx = \int x\left(1-\frac{2}{x^2+1}\right) dx$$

$$= \int x - \frac{2x}{x^2+1}\, dx = \frac{1}{2}x^2 - \ln(x^2+1) + c$$